ECE 488 – Automatic Control

Transfer Functions – Block Diagram Simplification

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Topics

- Linear system modeling (DC-motor)
- Linear time-invariant ordinary differential equations
- Block diagram representation

This Week

- Transfer functions
- Block diagram simplification

Transfer Functions: Transfer Block Representation

System Operator S

- Mapping from input signal u to output signal y: $y(t) = S(u(t))$

Gap 1

Transfer Function $G(s)$

- Relation between Laplace transforms (uni-directional) of input signal $u(t)$ and output signal $y(t)$

Transfer Functions: Laplace Domain Representation

Laplace Transform

- Transformed input signal: $u(t) \circ\!\!-\!\!\bullet\, \mathcal{L}(u(t)) = U(s)$
- Transformed output signal: $y(t) \circ\!\!-\!\!\bullet\, \mathcal{L}(y(t)) = Y(s)$
- Laplace transform of the LTI ODE (zero initial conditions):

$$\mathcal{L}\big(a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y\big) = \mathcal{L}\big(b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u\big)$$

$$\Big(\frac{d}{dt}u(t) \circ\!\!-\!\!\bullet\, sU(s) + u(0)\Big) \quad \Downarrow \quad \Big(\frac{d}{dt}y(t) \circ\!\!-\!\!\bullet\, sY(s) + y(0)\Big)$$

Gap 2

Transfer Functions: Input/Output Relation

Gap 3

Input/Output Relation: Transfer Function $G(s)$

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} =: G(s) = \frac{B(s)}{A(s)}$$

Transfer Block Representation

- Input/output variables u, y
- Dynamic relation G

Transfer Functions: Examples

RLC-Circuit

Gap 4

Transfer Functions: Example

DC-motor with "Transfer Function" Transfer Blocks

Gap 5

Transfer Functions: Facts

Impulse Response

- Consider system operator S and dirac impulse $\delta(t)$ (recall that $y(t) = S(u(t)))$
  $\Rightarrow$ System reaction to Dirac impulse input signal $u(t) = \delta(t)$

$$g(t) = S(\delta(t))$$

  $\Rightarrow$ It holds that $g(t) \circ\!\!-\!\!\bullet\, G(s)$

Output Computation

- Convolution: $g(t) \star u(t) = \int_{-\infty}^{\infty} g(t-\tau)\, u(\tau)\, d\tau$

$$\Rightarrow y(t) = g(t) \star u(t)$$

- Inverse Laplace transform: $y(t) = \mathcal{L}^{-1}(Y(s))$

$$\Rightarrow y(t) = g(t) \star u(t) = \mathcal{L}^{-1}(G(s)\, U(s)) = \mathcal{L}^{-1}(Y(s))$$

Step Responses: Computation

Definition

The unit step response is the output response of a dynamic system to the Heaviside step function applied at its input

- Heaviside step function: $\sigma(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$

Step Response Computation

- Input: $u(t) = \sigma(t) \circ\!\!-\!\!\bullet\ U(s) = \frac{1}{s}$
- Output: $Y(s) = G(s)U(s) = G(s)\frac{1}{s} \bullet\!\!-\!\!\circ\ y(t) = \mathcal{L}^{-1}\big(G(s)\frac{1}{s}\big)$

Gap 6

Step Responses: Integrator Example

Integrator

- ODE: $\dot{y} = u$
- Laplace transform: $sY(s) = U(s)$
- Transfer function: $G(s) = \frac{1}{s}$
- Step response: $Y(s) = G(s)\frac{1}{s} = \frac{1}{s^2} \bullet\!\!-\!\!\circ\ \sigma(t) \cdot t$

Gap 7

Step Responses: First-order Lag Example

First-order Lag

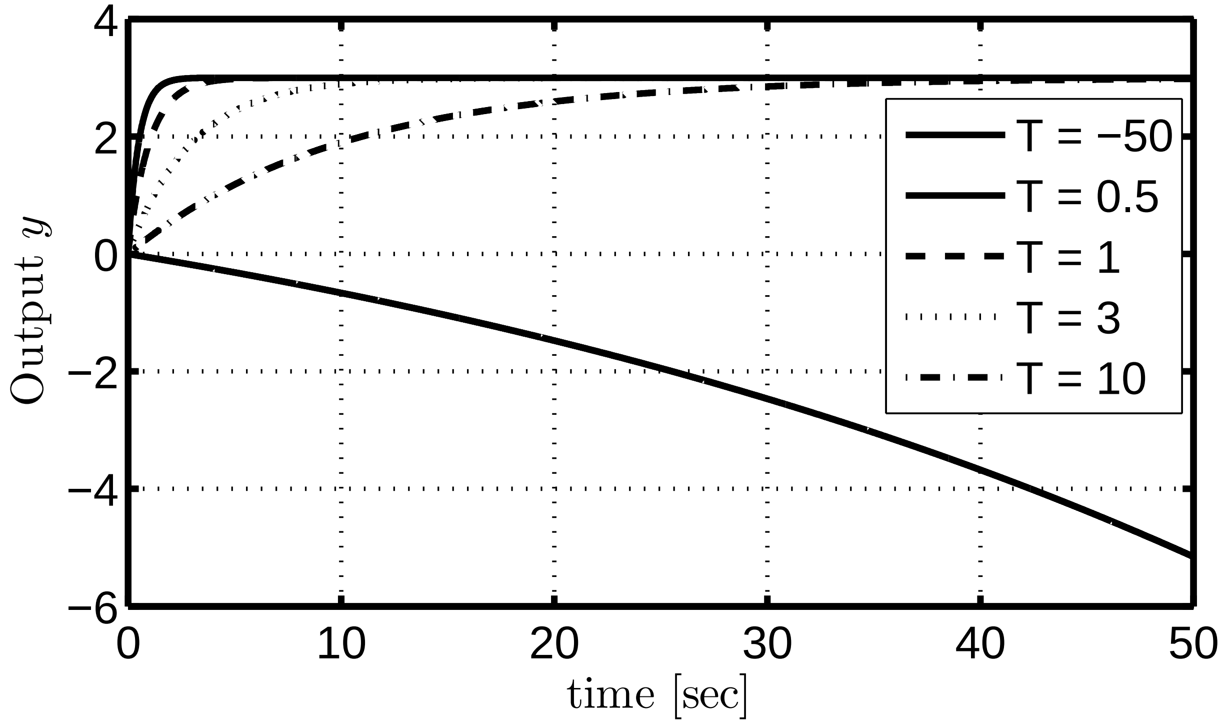
- ODE: $T \cdot \dot{y} + y = K \cdot u$
- Laplace transform:
- Transfer function:
- Step response:

Gap 8

Gap 9

Step Responses: First-order Lag Example

First-order Lag for Different T



Step Responses: Time Delay Example

Time Delay

- Input-output relation: $y(t) = u(t - \Delta)$

Gap 10

- Laplace transform:
- Transfer function:
- Step response:

Gap 11

Step Responses: Second-order Lag Example

Second-order Lag

- ODE: $T^2 \cdot \ddot{y} + 2DT\dot{y} + y = K \cdot u$
- Laplace transform: $s^2T^2 \cdot Y(s) + 2DTsY(s) + Y(s) = K \cdot U(s)$
- Transfer function: $G(s) = \dfrac{K}{1 + 2DT \cdot s + T^2 \cdot s^2}$
- Step response: Different solutions depending on damping constant D

Denominator Zeros

Gap 12

Step Responses: Second-order Lag Example

Case 1: $D > 1$ **(aperiodic case)**

- $G(s) = \dfrac{K}{(1+T_1 s)(1+T_2 s)}$
- $y(t) = \sigma(t) K (1 - \frac{T_1}{T_1 - T_2} e^{-t/T_1} + \frac{T_2}{T_1 - T_2} e^{-t/T_2})$

Case 2: $D = 1$ **(aperiodic limit)**

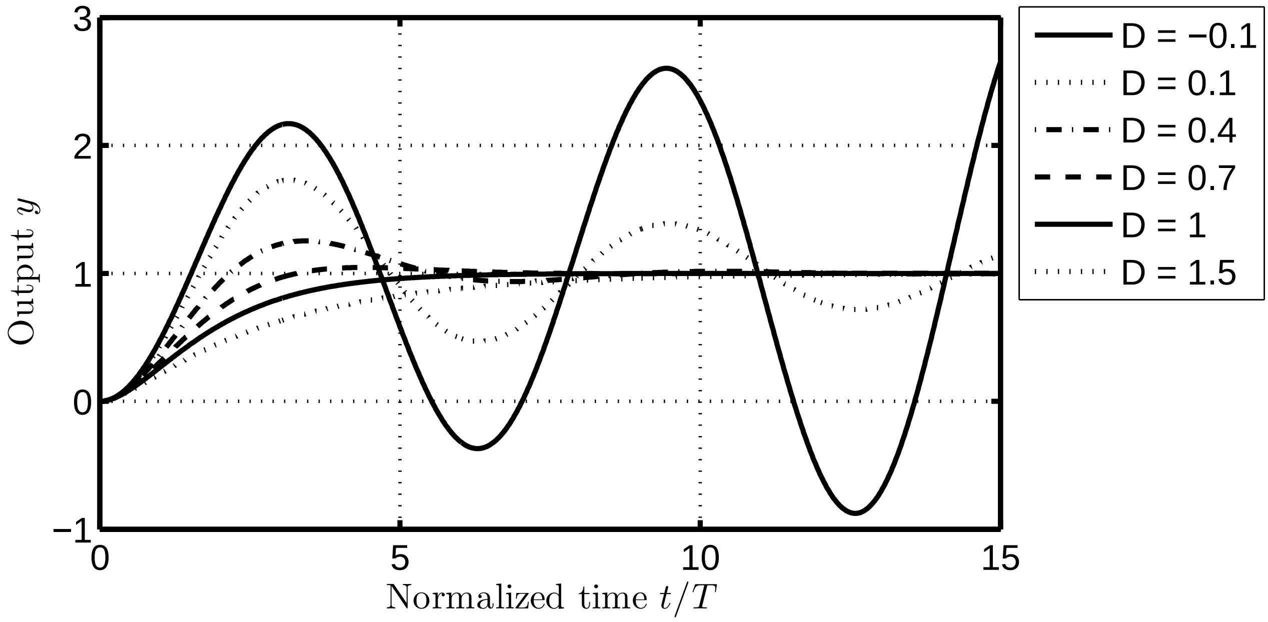
- $G(s) = \dfrac{K}{(1+Ts)^2}$
- $y(t) = \sigma(t) K \big(1 - (1 + \frac{t}{T}) e^{-t/T}\big)$

Case 3: $D < 1$ **(periodic case)**

- $G(s) = \dfrac{K}{(1 + 2DTs + T^2 s^2)}$
- $y(t) = \sigma(t) K \big(1 - \frac{e^{-Dt/T}}{\sqrt{1-D^2}} \sin(\frac{\sqrt{1-D^2}}{T} t + \arctan \frac{\sqrt{1-D^2}}{D})\big)$

Step Responses: Second-order Lag Example

Second-order Lag for Different D



Block Diagrams: Standardized Transfer Blocks

Proportional Gain

$$y(t) = K\sigma(t)$$

Integrator

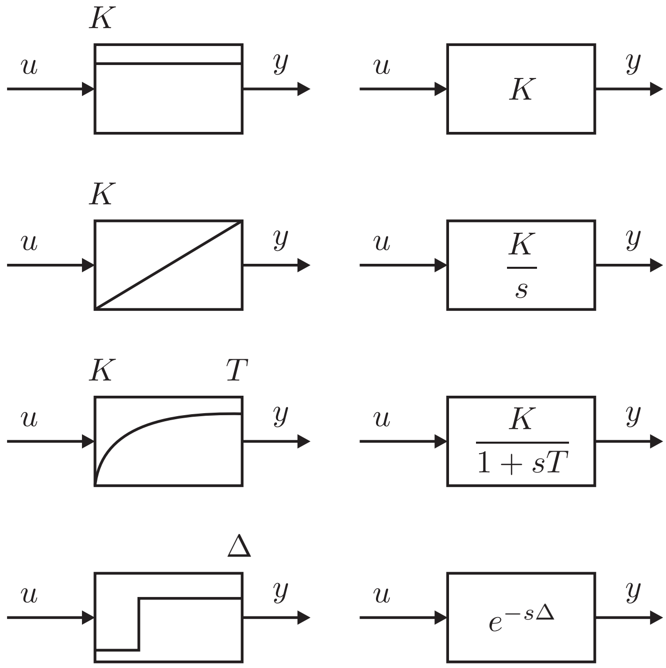
$$y(t) = K\sigma(t)t$$

First-order Lag

$$y(t) = K\sigma(t)(1 - e^{-t/T})$$

Time Delay

$$y(t) = \sigma(t - \Delta)$$



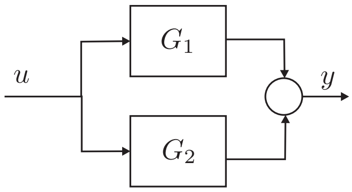
Plant Modeling: Example

DC-motor with "Step Response" Transfer Blocks

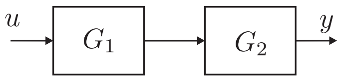
Gap 13

Block Diagram Simplification: Connection Rules

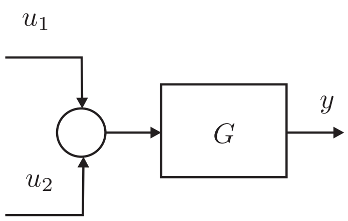
Parallel



Series



Summation 1

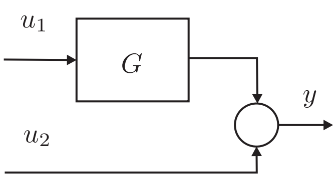


Equivalent Representation

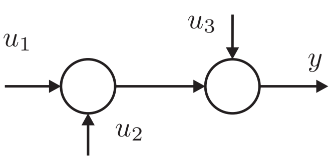
Gap 14

Block Diagram Simplification: Connection Rules

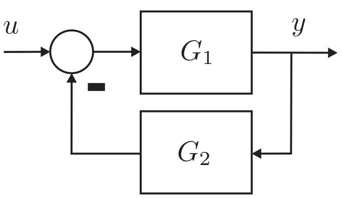
Summation 2



Summation 3



Feedback



Equivalent Representation

Gap 15

Block Diagram Simplification: Example

DC-motor Simplification

Gap 16

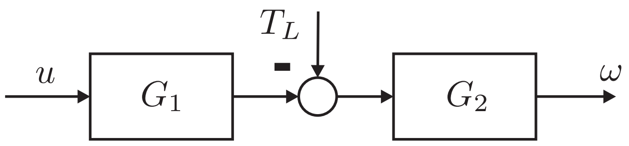
Block Diagram Simplification: Example

DC-motor Simplification

Gap 17

Block Diagram Simplification: Result

DC-Motor



Input-Output Behavior

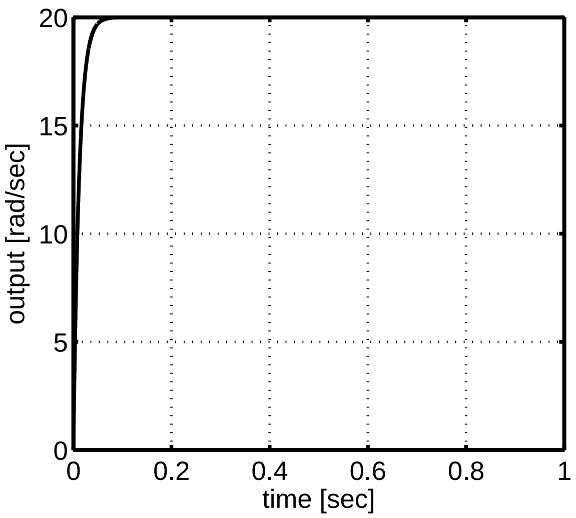
$$\frac{\Omega(s)}{U(s)} = G_1(s)G_2(s)$$

Disturbance-Output Behavior

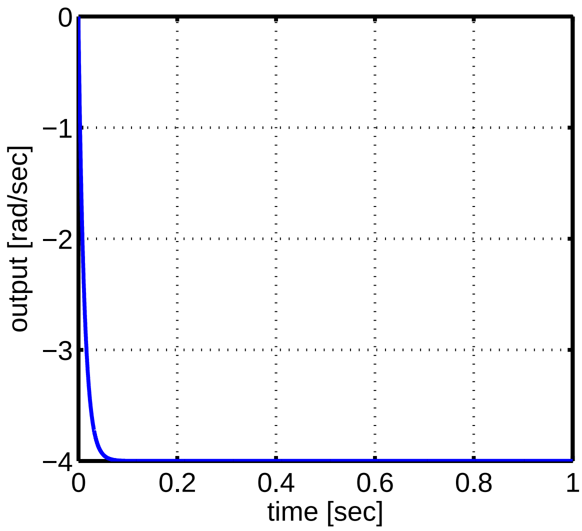
$$\frac{\Omega(s)}{T_L(s)} = G_2(s)$$

Block Diagram Simplification: Simulation

Input-Output Behavior



Disturbance-Output Behavior



Parameters

- $J_a = 3 \cdot 10^{-6}\,\mathrm{kg\,m^2}$; $R_a = 10\,\Omega$,
- $L_a = 2\,\mathrm{mH}$, $c\Phi_F = 0.05\,\mathrm{N\,m/A}$

Steps

- Input: voltage step of 1 V
- Disturbance: torque 10^{-3} Nm