ECE 488 – Automatic Control

State-Space Models – Nonlinear Modeling

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Topics

- Linear system modeling
- Block diagram representation
- Transfer functions
- Block diagram simplification

This Week

- State space models as alternative/equivalent system representation
- Nonlinear system modeling
- Set-point linearization of nonlinear state space models

Linear State Space Models: Definitions

Previously

- Study of dynamic relation between input (u) and output (y) signals

System State

The state $x(t_0)$ of a dynamic system at time t_0 is the information at time t_0 that is needed together with the input signal $u(t)$ for $t \geq t_0$ to determine the output signal $y(t)$ for $t \geq t_0$

Example

Gap 1

Linear State Space Models: Definitions

State Space Equations

$$\dot{x}(t) = A\,x(t) + b\,u(t) + o\,w(t)$$
$$y(t) = c^T\,x(t) + d\,u(t)$$

Signals

- State vector: $x(t) \in \mathbb{R}^n$
- State vector derivative: $\dot{x}(t) \in \mathbb{R}^n$
- Input: $u(t) \in \mathbb{R}$
- Output: $y(t) \in \mathbb{R}$
- Disturbance: $w(t) \in \mathbb{R}$

Constant Matrices and Vectors

- Dynamics matrix: $A \in \mathbb{R}^{n\times n}$
- Input vector: $b \in \mathbb{R}^{n\times 1}$
- Disturbance vector: $o \in \mathbb{R}^{n\times 1}$
- Output vector: $c^T \in \mathbb{R}^{1\times n}$
- Feed-through: $d \in \mathbb{R}$

Linear State Space Models: RLC-Circuit Example

Equations

Gap 2

Linear State Space Models: DC-Motor Example

Equations

Gap 3

Linear State Space Models: Relation to Transfer Function

Linear State Space Model

$$\dot{x} = A \cdot x + b \cdot u + o \cdot w$$
$$y = c^T \cdot x + d \cdot u$$

Laplace Transformation

$$sX(s) - x(0) = A \cdot X(s) + b \cdot U(s) + o \cdot W(s)$$
$$Y(s) = c^T \cdot X(s) + d \cdot U(s)$$

Computation

Gap 4

$$\Rightarrow Y(s) = \underbrace{(c^T(sI - A)^{-1}b + d)}_{\text{Plant transfer function } G(s)} U(s) + \underbrace{c^T(sI - A)^{-1}o}_{\text{Disturbance transfer function } G_d(s)} W(s)$$

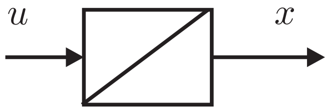
Linear State Space Models: RLC-Circuit Transfer Function

Computation

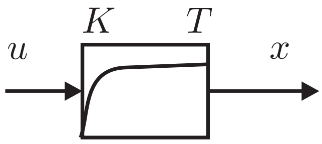
Gap 5

Linear State Space Models: Relation to Block Diagram

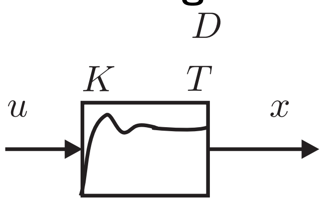
Integrator



First-order Lag



Second-order Lag



State Space Model

- State: x (integrator output)
- State equation: $\dot{x} = u$

State Space Model

- State: x (first-order lag output)
- State equation: $\dot{x} = \frac{1}{T}(K \cdot u - x)$

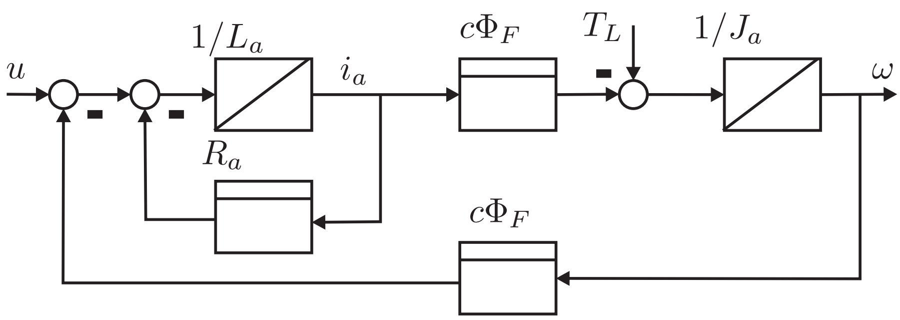
State Space Model

- States: x, $\hat{x}$
- State equation:

$$\dot{x} = \hat{x}$$

$$\dot{\hat{x}} = \frac{1}{T^2}(K \cdot u - x - 2\,D\,T\,\hat{x})$$

Linear State Space Models: DC-Motor Example



Computation

Gap 6

Nonlinear System Modeling: Remarks

LTI System Operators

- Proportional gain
- Differentiation
- Integration
- Lead/lag components
- Summations

⇒ All linear operators can be represented by transfer functions

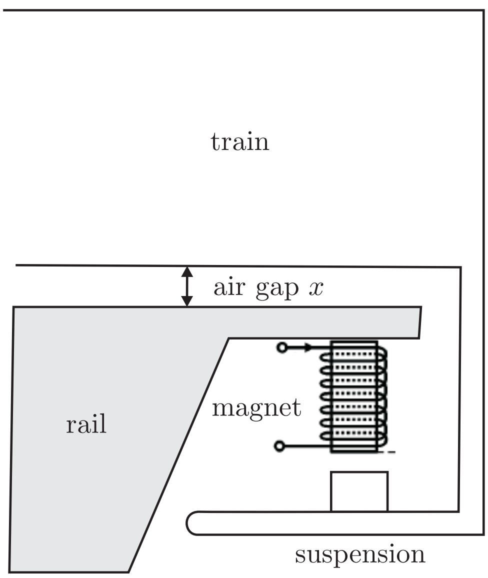
Nonlinear Systems

- Contain nonlinear system operators

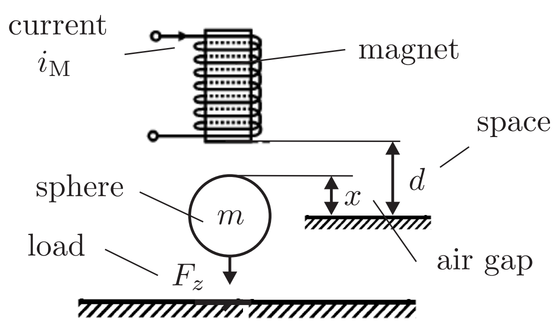
⇒ No transfer function representation

Nonlinear System Modeling: Magnetic Suspension

Schematic



Simplified Description



Simplifications

- Sphere represents vehicle
- Magnet represents suspension system

Nonlinear System Modeling: Equations

Computation

Gap 7

Nonlinear State Equations: General Form

State Equations

$$\dot{x} = f(x, u, w)$$
$$y = h(x, u)$$

Notation

- state: x, output: y, input: u, disturbance w
- f: continuous in x, u, w and additional assumptions (see for example ECE 564)
- h: continuous in x, u

Example

Gap 8

Nonlinear State Equations: Block Diagram

Example

Gap 9

Nonlinear System Modeling: Remarks

Synthesis and Analysis Techniques for Nonlinear Systems

- Beyond the scope of this lecture
  $\rightarrow$ Master-level course ECE 564
- Extensive literature
  $\rightarrow$ Alberto Isidori: "Nonlinear Control Systems", Springer, 1995 (ISBN: 3-54-019916-0)
  $\rightarrow$ Hassan K. Khalil: "Nonlinear Systems", Prentice Hall, 2002 (ISBN: 0-13-067389-7)

Set-point Linearization

- Consider system behavior in the vicinity of a given set-point
  $\rightarrow$ Assume almost linear behavior close to the set-point
  $\rightarrow$ Find a linear system model to approximate the nonlinear system

Set-Point: Definition

Set-point Definition

A set point is a stationary (non-changing) state of a system where the system output maintains a constant set-point value y_{SP}

Computation of a Set-point

- Given: y_{SP}, w_{SP}
- We want to compute x_{SP} (constant set-point value of the state) and u_{SP} (constant set-point value of the input)
- Computation

$$y_{SP} = h(x_{SP}, u_{SP})$$
$$0 = \dot{x} = f(x_{SP}, u_{SP}, w_{SP})$$

$\Rightarrow$ Solve for x_{SP}, u_{SP}

Set-Point: Example

Magnetic Suspension

Gap 10

Set-point Linearization: Description

Explanation

- Compute a „small signal“ approximation of the nonlinear system that is valid close to the set-point
- Introduce „Difference variables“ (deviation from the set-point)
- $\Delta x = x - x_{SP}$, $\Delta y = y - y_{SP}$, $\Delta u = u - u_{SP}$, $\Delta w = w - w_{SP}$

Taylor Series Expansion

$$\Delta\dot{x} = \dot{x} \approx \underbrace{f(x_{SP}, u_{SP}, w_{SP})}_{=0} + \underbrace{\frac{\partial f}{\partial x}|_{SP}}_{A} \Delta x + \underbrace{\frac{\partial f}{\partial u}|_{SP}}_{b} \Delta u + \underbrace{\frac{\partial f}{\partial w}|_{SP}}_{o} \Delta w$$

$$= A\,\Delta x + b\,\Delta u + o\,\Delta w$$

$$\Delta y \approx \underbrace{h(x_{SP}, u_{SP}) - y_{SP}}_{=0} + \underbrace{\frac{\partial h}{\partial x}|_{SP}}_{c^T} \Delta x + \underbrace{\frac{\partial f}{\partial u}|_{SP}}_{d} \Delta u = c^T\,\Delta x + d\,\Delta u$$

Set-point Linearization: Example

Example Equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -g + \frac{K_M}{m}\frac{u^2}{(d - x_1)^2} - \frac{1}{m}\,w$$

$$y = x_1$$

Computation

Gap 11

Set-point Linearization: Example

Computation

Gap 12

Set-point Linearization: Illustration

Tangent Approximation

- Compute slope of nonlinear function in set-point
  $\rightarrow$ Replace nonlinear function by its tangent at set-point

Illustration

Gap 13

Set-point Linearization: Magnetic Suspension Example

Linearized Block Diagram

Gap 14

Linearization: Summary

Task

- Characterize nonlinear system behavior close to a set-point

Method

- Write system representation in terms of “difference variables”
- Use first-order Taylor series approximation for nonlinearities

Result

- We get a linear system model for the nonlinear system
- Linear methods can be used for the nonlinear system close to the set-point
- Important restriction
  $\rightarrow$ Linear model is only valid in the vicinity of the set-point