ECE 488 – Automatic Control

Nyquist Plot

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Weeks

- Plant modeling
- Properties of transfer functions
- Feedback loop
  - Sensitivity transfer functions
  - Internal stability
- Root locus method

This week

- Nyquist plot
- Nyquist criterion

Stability Analysis: Reminder

Basic Feedback Control Loop

Gap 1

- The feedback loop is internally stable if all zeros of $1 + C(s)\,G(s)$ lie in the OLHP
  $\Rightarrow$ Direct computation of the zeros of $1 + C(s)\,G(s)$
  $\Rightarrow$ Analytical verification using Routh-Hurwitz test
  $\Rightarrow$ Graphical verification based on root locus plot
  $\Rightarrow$ This week: Graphical verification using Nyquist plot

Nyquist Plot: Frequency Response

Open Loop Properties

$$G_o(s) = C(s) \cdot G(s) = K\,\frac{(s-z_1)\,(s-z_2)\,\cdots\,(s-z_m)}{(s-p_1)\,(s-p_2)\,\cdots\,(s-p_n)}$$

- $G_o(s)$ has poles $p_1, \cdots, p_n$
- $G_o(s)$ is a complex number for all $s \in \mathbb{C}$

Example

Gap 2

$\Rightarrow$ **Plot $G_o(s)$ in the complex plane along a well-defined path for s**

Nyquist Plot: Construction

Closed Path $\mathcal{C}$ in Complex Plane

- $\mathcal{C}$ includes the imaginary axis
- $\mathcal{C}$ encircles all poles of G_o on the imaginary axis by a small semi-circle in the right-half plane
- $\mathcal{C}$ closes by a large semi-circle in the right-half plane
- $\mathcal{C}$ is traversed in clockwise direction
  $\Rightarrow$ **Such path $\mathcal{C}$ encircles all poles of $G_o(s)$ in the right-half plane**

Illustration: s-Plane

Gap 3

Nyquist Plot: Path Representation

Components of the Closed Path $\mathcal{C}$

- Imaginary axis: $s = j\omega$ for $-\infty \leq \omega \leq \infty$
- Large semi-circle: $s = R \cdot e^{j\Phi}$ for $\pi/2 \geq \Phi \geq -\pi/2$ and R large
- Small semi-circles for each p_i on imaginary axis: $s = p_i + r \cdot e^{j\varphi}$ for $-\pi/2 \leq \varphi \leq \pi/2$ and r small

Nyquist Curve

- Defined by mapping $s \mapsto G_o(s)$ for all $s \in \mathcal{C}$
  $\Rightarrow$ Each $s \in \mathcal{C}$ is mapped to the complex number $G_o(s)$

Nyquist Plot

- Graphical representation of the Nyquist curve in the complex plane
  $\rightarrow$ We call the complex plane G_o-plane
  $\rightarrow$ Complex number $G_o(s)$ is plotted in the G_o-plane for each $s \in \mathcal{C}$

Nyquist Plot: Assumptions

Assumptions for $G_o(s)$

$$G_o(s) = K\,\frac{(s-z_1)\cdots(s-z_m)}{s^q(s-p_{q+1})\cdots(s-p_n)}$$

- $G_o(s)$ is proper: $r = n - m \geq 0$
- $G_o(s)$ has $0 \leq q \leq n$ poles at zero and $n-q$ poles different from zero

Example

Gap 4

Nyquist Plot: Construction

Large Semi-Circle: $s = R\cdot e^{j\Phi}$, $\pi/2 \geq \Phi \geq -\pi/2$

$$\lim_{R\to\infty} G_o(R\cdot e^{j\Phi}) \approx K\cdot\frac{R^m e^{jm\Phi}}{R^n e^{jn\Phi}} = K\cdot\frac{1}{R^r}e^{-jr\Phi} = \begin{cases} 0 & \text{if } r>0 \\ K & \text{if } r=0 \end{cases}$$

$\Rightarrow$ Maps to point on real axis in complex plane

Example: $G_o(s) = \dfrac{1}{s\,(s+1)}$

Gap 5

Nyquist Plot: Construction

Small Semi-Circles for poles at zero: $s = r \cdot e^{j\varphi}$**,** $-\pi/2 \leq \varphi \leq \pi/2$

$$G_o(r \cdot e^{j\varphi}) \approx K \cdot \frac{(-z_1)\cdots(-z_m)}{r^q e^{jq\varphi}(-p_{q+1})\cdots(-p_n)} = K_r \cdot \frac{1}{r^q} \cdot e^{-jq\varphi}$$

$\Rightarrow$ Maps to a curve with large radius that encircles the origin $q/2$ times
$\Rightarrow$ Circle starts at $\angle(K_r) + q\,\pi/2$ and ends at $\angle(K_r) - q\,\pi/2$
$\Rightarrow$ Direction is clockwise

Example: $G_o(s) = \dfrac{1}{s\,(s+1)}$

Gap 6

Reminder Stability Analysis Nyquist Plot

Nyquist Plot: Construction

Imaginary Axis: $s = j\omega$**,** $-\infty < \omega < \infty$

$$G_o(j\omega) = K\,\frac{(j\omega - z_1)\cdots(j\omega - z_m)}{(j\omega)^q(j\omega - p_{q+1})\cdots(j\omega - p_n)} =$$

$$= K_r\,\frac{(1 - j\omega/z_1)\,(1 - j\omega/z_2)\,\cdots\,(1 - j\omega/z_m)}{(j\omega)^q\,(1 - j\omega/p_1)\,(1 - j\omega/p_2)\,\cdots\,(1 - j\omega/p_n)}$$

$$K_r = K\,\frac{(-z_1)\,(-z_2)\,\cdots\,(-z_m)}{(-p_1)\,(-p_2)\,\cdots\,(-p_n)}$$

Limit for $\omega \to 0$

$$G_o(j\omega) \approx \frac{K_r}{(j\omega)^q}$$

$\Rightarrow$ Phase: $\lim_{\omega\to 0} \angle(G_o(j\omega)) = \angle(K_r) - q\,\dfrac{\pi}{2}$

$\Rightarrow$ Magnitude: $\lim_{\omega\to 0} |G_o(j\omega)| = \begin{cases} \infty & \text{if } q > 0 \\ |K_r| & \text{if } q = 0 \end{cases}$

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Nyquist Plot: Construction

Imaginary Axis: $s = j\omega$, $-\infty < \omega < \infty$

$$G_o(j\omega) = K\,\frac{(j\omega - z_1)\cdots(j\omega - z_m)}{(j\omega)^q (j\omega - p_{q+1})\cdots(j\omega - p_n)}$$

Limit for $\omega \to \infty$

$$G_o(j\omega) \approx K\,\frac{(j\omega)^m}{(j\omega)^n} = \frac{K}{(j\omega)^{n-m}} = \frac{K}{(j\omega)^r}$$

$\Rightarrow$ Phase: $\lim_{\omega\to\infty} \angle(G_o(j\omega)) = \angle(K) - r\,\frac{\pi}{2}$

$\Rightarrow$ Magnitude $\lim_{\omega\to\infty} |G_o(j\omega)| = \begin{cases} 0 & \text{if } r > 0 \\ |K| & \text{if } r = 0 \end{cases}$

Limit for $\omega \to -\infty$

$\Rightarrow$ Symmetric to real axis compared to $\omega \to \infty$

Nyquist Plot: Example

Example: $G_o(s) = \dfrac{1}{s\,(s+1)}$

Gap 7

Nyquist Plot: Example

Plot

Gap 8

Nyquist Plot: Example

Example: $G_o(s) = -\dfrac{s+2}{(s+4)\,(s^2+2\,s+2)}$

Gap 9

Nyquist Plot: Example

Plot

Gap 10

Nyquist Plot: Example

Example: $\dfrac{10\,s+3}{s^2+4\,s+4}$

Gap 11

Nyquist Plot: Example

Plot

Gap 12

Nyquist Plot: Stability

Nyquist Criterion

- Consider the basic feedback loop with open-loop transfer function $G_o(s) = C(s)G(s)$

  Gap 13

- Call P the number of poles of G_o in the open right-half plane
- Call N the number of times the Nyquist plot of G_o encircles the point $(-1, 0)$ in clockwise direction
  $\Rightarrow$ The closed loop is internally stable if and only if $N + P = 0$

Nyquist Plot: Stability Analysis Example

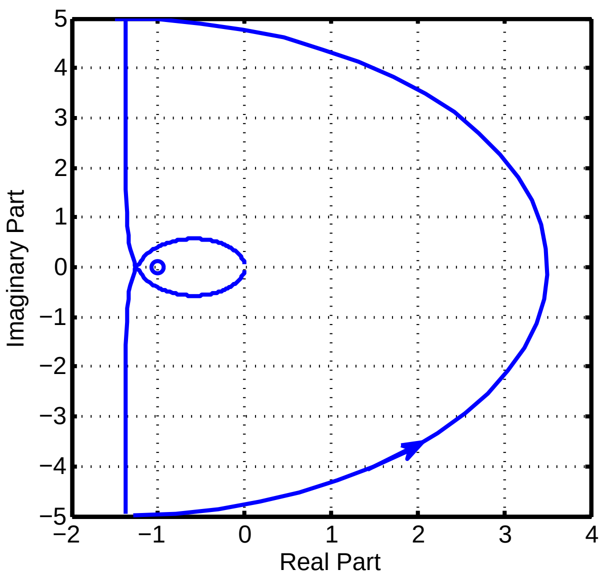
Example

Gap 14

Nyquist Plot: Stability Analysis Example

Instable Open Loop

- $G_o(s) = \dfrac{5s+2}{s(s-4)}$



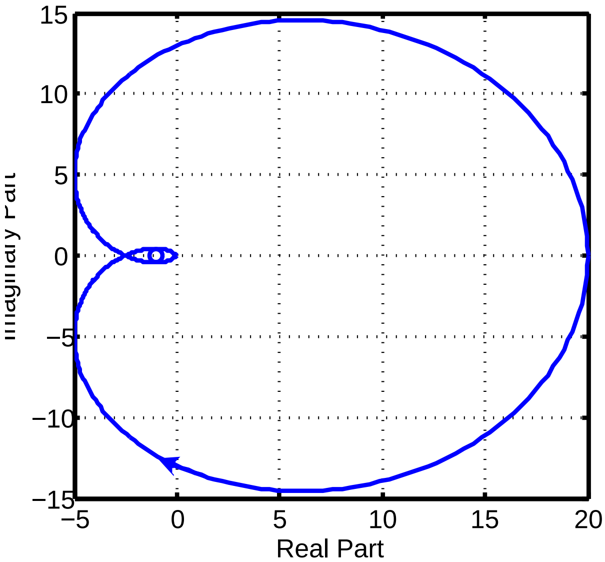
Computation

Gap 15

Nyquist Plot: Stability Analysis Example

Stable Open Loop

- $G_o(s) = \dfrac{20}{(1+s)^3}$



Computation

Gap 16

Nyquist Plot: Gain Margin

Assumption

- Open loop transfer function $G_o(s)$ without poles in the open right half plane

Gain Margin

- Multiplication of G_o with constant K_g leads to instable closed loop
- Phase crossover frequency ω_p such that $\angle G_o(\omega_p) = -\pi$
  $\rightarrow$ Gain margin K_g describes degree of stability with respect to gain changes

Illustration

Gap 17

Nyquist Plot: Phase Margin

Assumption

- Open loop transfer function $G_o(s)$ without poles in the open right half plane

Phase Margin

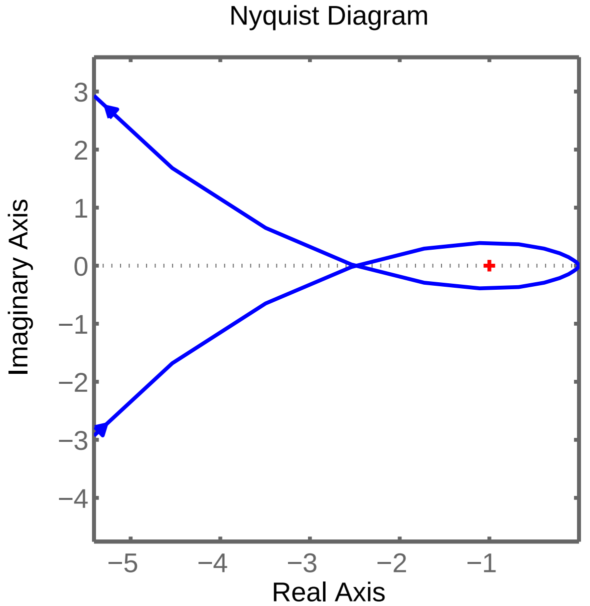
- Multiplication of G_o with $e^{-j\Phi_m}$ (phase shift of Φ_m) leads to instable closed loop
- Gain crossover frequency ω_g such that $|G_o(\omega_g)| = 1$
  $\rightarrow$ Phase margin Φ_m describes degree of stability with respect to phase shift

Illustration

Gap 18

Nyquist Plot: Example

Example: $G_o(s) = \dfrac{0.005\,(1+s)}{s\,(1+1000\,s)^2}$



Computation

Gap 19