ECE 488 – Automatic Control

Bode Plot

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Weeks

- LTI system modeling
- Nonlinear modeling and linearization
- Stability
- Steady-state and transient response
- Feedback Control
  - Root locus
  - Nyquist plot

This week

- Frequency response
- Bode plot

Frequency Response: Basic Idea

Given

- Stable LTI system with transfer function $G(s)$

Goal

- Find system response $y(t)$ for sinusoidal input signal

$$u(t) = \sin(\omega t)$$

Gap 1

Solution

- Consider output computation in the Laplace domain: $Y(s) = G(s)\,U(s)$
- Sinusoidal input function: $U(s) = \frac{\omega}{s^2 + \omega^2}$

Frequency Response: Output Computation

Output Computation

Gap 2

Frequency Response: Steady State Response

Intermediate Result

$$Y_{ss}(s) = \frac{G(j\omega)}{2j}\frac{1}{s - j\omega} + \frac{G(-j\omega)}{-2j}\frac{1}{s + j\omega}$$

Computation of the Steady State Response

Gap 3

Frequency Response: Result

Result

$$y(t) = |G(j\omega)| \sin\left(\omega t + \angle(G(j\omega))\right)$$

Description

- Output signal y oscillates with same frequency ω as input signal u
- Amplification of u by $|G(j\omega)|$
- Phase shift of u by $\angle(G(j\omega))$

Illustration

Gap 4

Bode Plot: Basic Idea

Description

- Given: Transfer function $G(s)$
- Task: Show the frequency response in terms of magnitude $|G(j\omega)|$ and phase shift $\angle(G(j\omega))$

Magnitude Plot

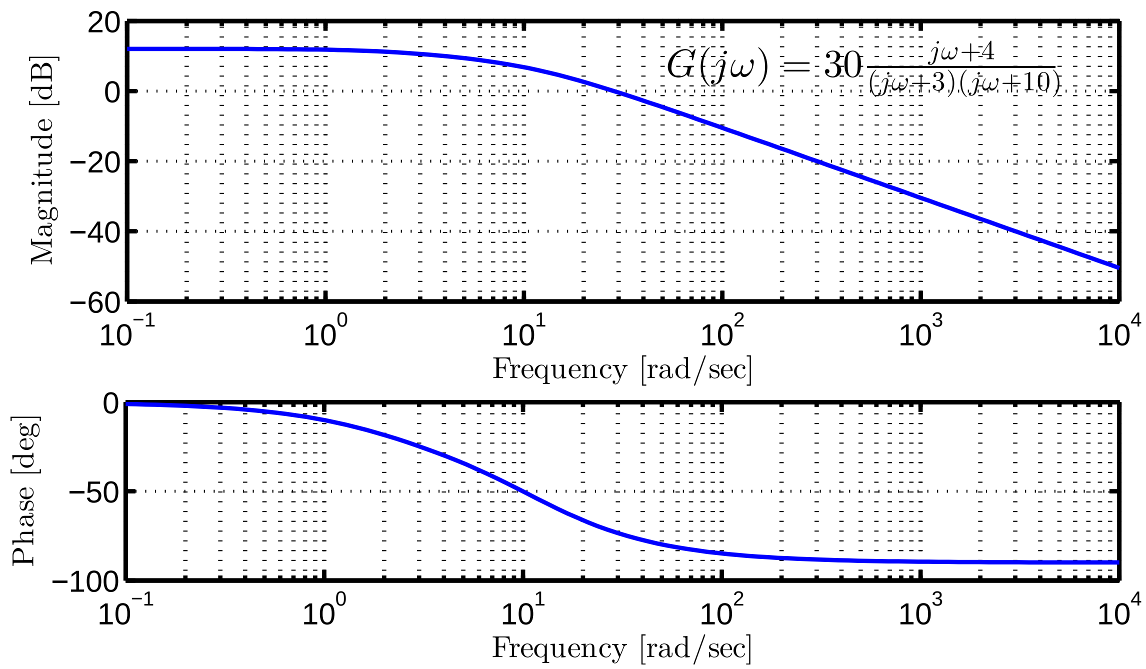
- Frequency axis with logarithmic scale ω [rad/sec]
- Magnitude axis with $20 \log |G(j\omega)|$ [dB]

Phase Plot

- Frequency axis with logarithmic scale ω [rad/sec]
- Phase axis with $\angle G(j\omega) = \arctan(\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))})$ [°]

Bode Plot: Example

Bode Plot Example



Bode Plot: Transfer Function Representation

Time-constant Representation

$$G(s) = K_{\mathrm{DC}} \frac{(1+\tau_1\, s)(1+\tau_2\, s)\cdots(1+2\delta_f\tau_f\, s+\tau_f^2\, s^2)\cdots}{s^q(1+T_1\, s)(1+T_2\, s)\cdots(1+2D_g\, T_g\, s+T_g^2\, s^2)\cdots}$$

- Time constants for real zeros/poles: $\tau_1, \tau_2, \cdots$; T_1, $T_2, \cdots$
- Time constants for conjugated complex zeros/poles: $\tau_f, \cdots$; $T_g, \cdots$
- Damping for conjugated complex zeros/poles: $\delta_f, \cdots$; $D_g, \cdots$
- Multiplicity of pole at zero: q
  $\Rightarrow$ If the transfer function is not given in the time-constant representation, it has to be transformed to this representation

Frequency Response

$$G(s) = K_{\mathrm{DC}} \frac{(1+j\omega\tau_1)(1+j\omega\tau_2)\cdots(1+j\omega 2\delta_f\tau_f+(j\omega)^2\tau_f^2)\cdots}{(j\omega)^q(1+j\omega T_1)(1+j\omega T_2)\cdots(1+j\omega 2D_g T_g+(j\omega)^2 T_g^2)\cdots}$$

Bode Plot: Transfer Function Representation

Example

Gap 5

Standard Numerator/Denominator Factors

- K_{DC}
- s
- $1+T\,s$
- $1+2\,D\,T\,s+T^2\,s^2$

Bode Plot: Common Examples

DC Gain: $G(j\omega) = K_{DC}$

- Magnitude: $|G(j\omega)| = |K_{DC}| \Rightarrow |G(j\omega)|_{dB} = 20 \log K_{DC}$
- Phase $\angle(G(j\omega)) = \begin{cases} 0^\circ & \text{if } K_{DC} > 0 \\ 180^\circ & \text{if } K_{DC} < 0 \end{cases}$

Integrator $G(j\omega) = \dfrac{1}{j\omega}$

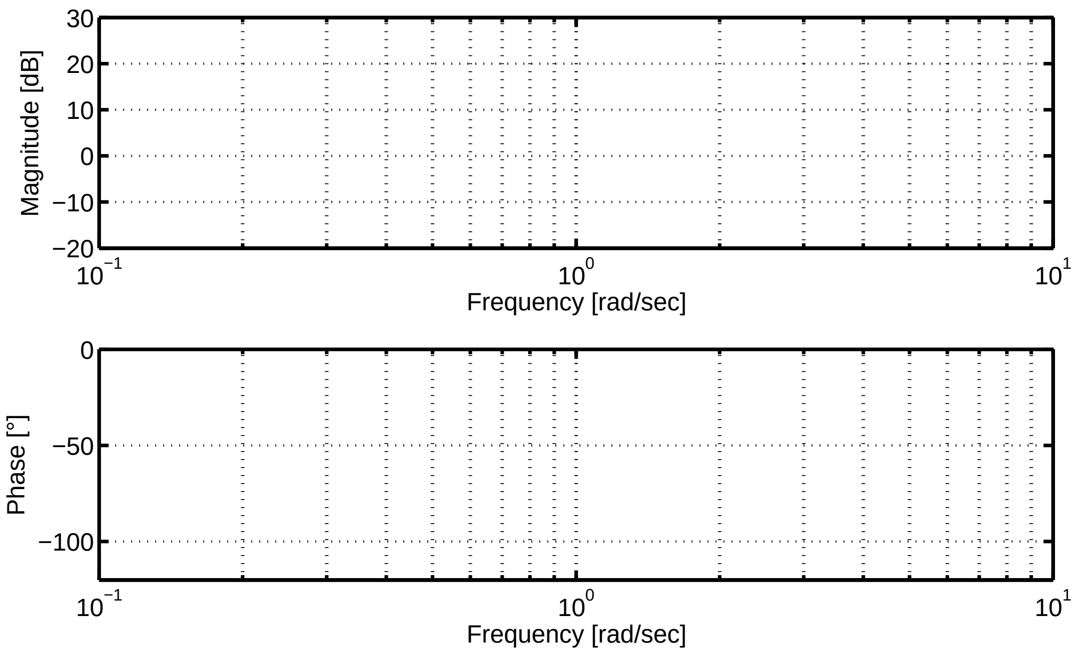
- Magnitude: $|G(j\omega)| = \dfrac{1}{\omega} \Rightarrow |G(j\omega)|_{dB} = -20 \log \omega$
- Phase: $\angle G(j\omega) = -90^\circ$

Combination of DC Gain and Integrator: $G(j\omega) = K_{DC} \dfrac{1}{j\omega}$

- Magnitude: $G(j\omega)|_{dB} = 20 \log K_{DC} - 20 \log \omega$
- Phase: $\angle(G(j\omega)) = \angle(K_{DC}) - 90^\circ$

Bode Plot: Common Examples

Bode Plot Construction: $G(s) = \dfrac{2}{s}$



Bode Plot: Examples

First-order Lag $G(j\omega) = \dfrac{1}{1 + j\omega T}$

- Magnitude: $|G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2 T^2}}$

$$\Rightarrow |G(j\omega)|_{dB} = 20\big(-1/2 \log(1+\omega^2 T^2)\big)$$

$$\approx \begin{cases} 0 & \omega < 1/T \\ -20 \log \omega T & \omega > 1/T \end{cases}$$

$\Rightarrow$ Straight-line approximation that bends at $\omega = 1/T$

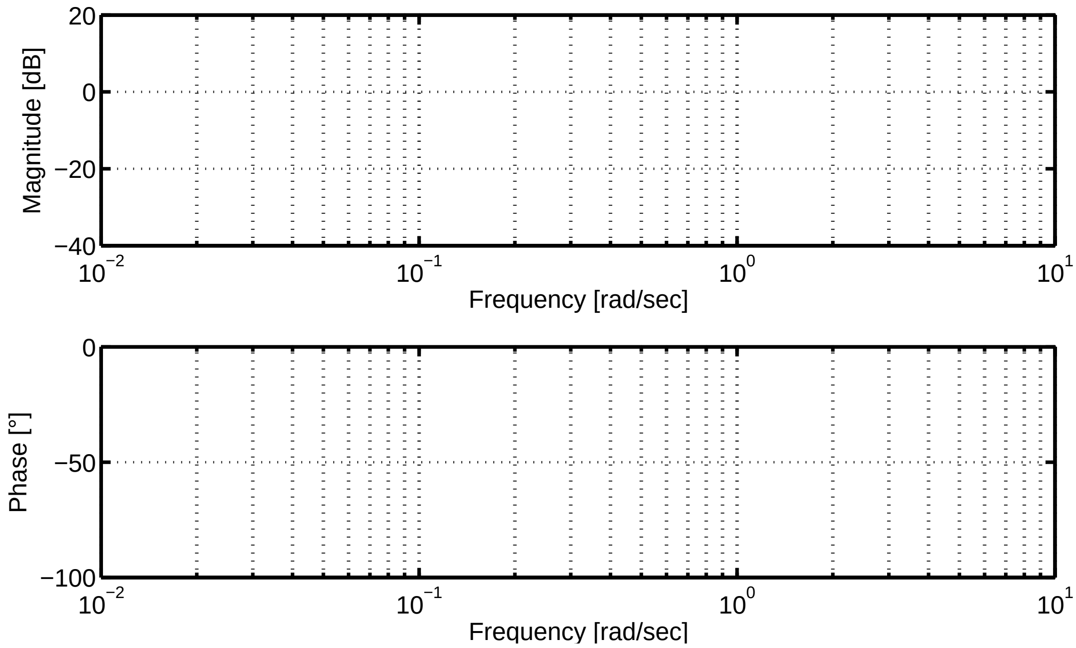
- Phase: $\angle G(j\omega) = -\angle(1 + j\omega T) = -\arctan \omega T$

$$\Rightarrow \angle G(j\omega) \approx \begin{cases} 0^\circ & \omega < 1/(10\,T) \\ -90^\circ & \omega > 10/T \end{cases}$$

$\Rightarrow$ Straight-line approximation that decreases from 0 to -90° between $\omega = 1/(10\,T)$ to $\omega = 10/T$

Bode Plot: First-order Lag

Bode Plot Construction: $G(s) = \dfrac{1}{1 + 4\,s}$



Bode Plot: Second-order Lag

Second-order Lag $G(j\omega) = \dfrac{1}{1 + 2DTj\omega + T^2(j\omega)^2}$

- Magnitude: $|G(j\omega)| = \dfrac{1}{\sqrt{(1 - T^2\omega^2)^2 + 4D^2T^2\omega^2}}$

$$\Rightarrow |G(j\omega)|_{dB} = 20\Big(-1/2 \log \big((1 - T^2\omega^2)^2 + 4D^2T^2\omega^2\big)\Big)$$

$$\approx \begin{cases} 0 & \omega < 1/T \\ -40 \log \omega T & \omega > 1/T \end{cases}$$

$\Rightarrow$ Straight-line approximation that bends at $\omega = 1/T$

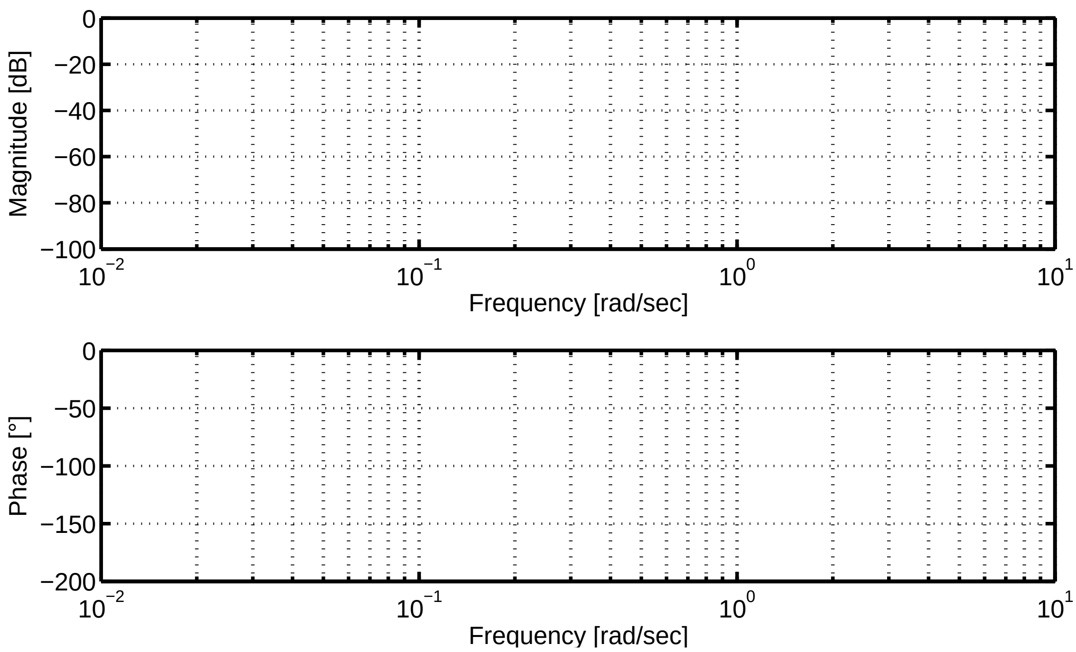
- Phase: $\angle G(j\omega) = -\angle(j2DT\omega + 1 - \omega^2T^2) = -\arctan \frac{2DT\omega}{1-\omega^2T^2}$

$$\Rightarrow \angle G(j\omega) \approx \begin{cases} 0^\circ & \omega << 1/(10\,T) \\ -180^\circ & \omega >> 10/T \end{cases}$$

$\Rightarrow$ Straight-line approximation that decreases from 0 to -180° between $\omega = 1/(10\,T)$ and $\omega = 10/T$

Bode Plot: Examples

Bode Plot Construction: $G(s) = \dfrac{1}{1 + 10\,s + 100\,s^2}$



Bode Plot: Multiplication Rules

Multiplication of Transfer Functions

$$G(s) = G_1(s) \cdot G_2(s) \cdot \cdots \cdot G_n(s)$$

Addition of Magnitude and Phase in Bode Plot

- $|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} + \cdots + |G_n(j\omega)|_{dB}$
- $\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \cdots + \angle G_n(j\omega)$

Gap 6

Bode Plot: Transfer Function Zeros

Inverse of Transfer Functions

$$G(s) = G_1^{-1}(s)$$

Negation of Magnitude and Phase

- $|G(j\omega)|_{dB} = -|G_1(j\omega)|_{dB}$
- $\angle G(j\omega) = -\angle G_1(j\omega)$

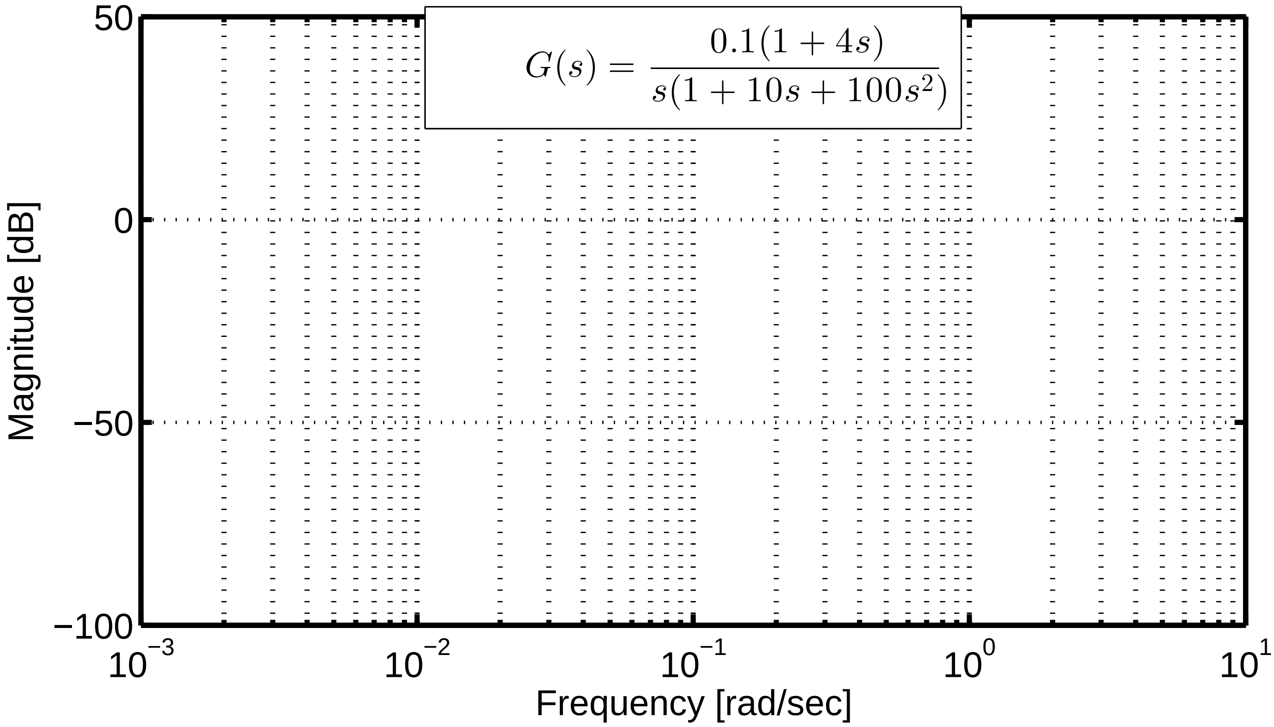
Gap 7

Bode Plot: Example

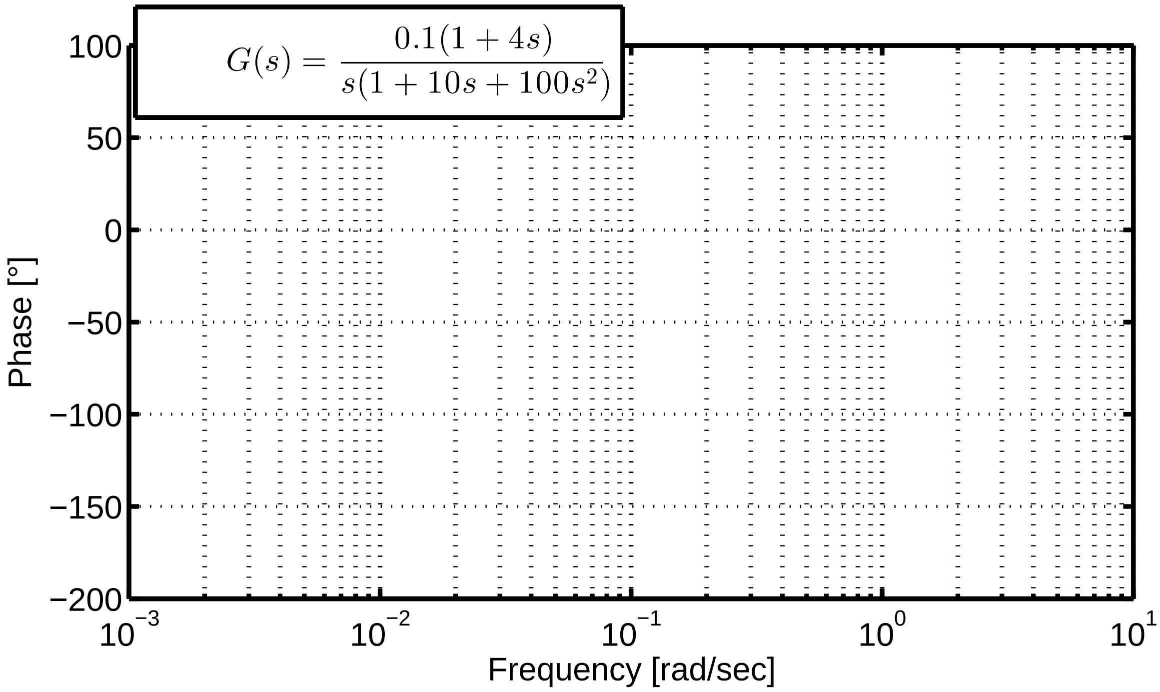
Computation

Gap 8

Bode Plot: Example



Bode Plot: Example



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Bode Plot: Non-minimum Phase Factors

First-order Lag Example

$$G(s) = \frac{1}{1 - T\,s}$$

Comparison to Minimum-Phase Factor

- $|G(j\omega)|_{dB} = |\frac{1}{1 + j\omega\,T}|_{dB}$; $\angle G(j\omega) = -\angle(\frac{1}{1 + j\omega\,T})$

Gap 9

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Bode Plot: Example

Computation

Gap 10

Bode Plot: Example $G(s) = \frac{1+0.1\,s}{1-s}$