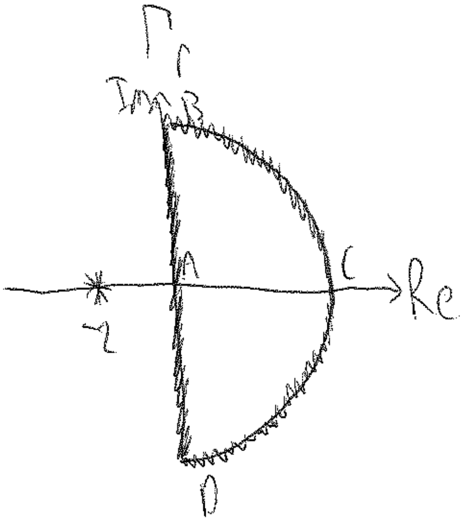
Exercise Questions ECE 488

Q-1) $H(s) = \dfrac{K}{(s+2)^2}$

Choose a contour in open RHP excluding singularities (s values that make $H(s) = \infty$).



$$\Gamma_r = \left[\begin{array}{l} AB: s = j\omega \quad \omega: 0 \to \infty \\ BC: s = Re^{j\theta} \quad R \to \infty \text{ and } \theta: 90 \to 0 \\ \text{other parts are symmetric} \\ CD \text{ is symmetric with } BC \\ \text{and } DA \text{ is symmetric wib } AB \end{array}\right]$$

To find the stability check $\dfrac{H(s)}{K} = \dfrac{1}{(s+2)^2}$

b/w AB $s = j\omega$

$$G(s)\Big|_{s=j\omega} = \frac{H(s)}{K}\Big|_{s=j\omega} = \frac{1}{(j\omega+2)^2} = \frac{1}{\underset{(4-\omega^2-4j\omega)}{4-\omega^2+4j\omega}} = \frac{4-\omega^2-4j\omega}{(4-\omega^2)^2+16\omega^2} = \frac{4-\omega^2}{(4-\omega^2)^2+16\omega^2} + \frac{-4j\omega}{(4-\omega^2)^2+16\omega^2}$$

$$Im\{G(j\omega)\} = \frac{-4\omega}{(4-\omega^2)^2+16\omega^2} \qquad Re\{G(j\omega)\} = \frac{4-\omega^2}{(4-\omega^2)^2+16\omega^2}$$

	~~$\omega=0$~~	~~$0<\omega<\infty$~~	~~$\omega=\infty$~~
~~Im(G(jω))~~	~~0~~	~~—~~	~~0~~
~~Re(G(jω))~~	~~1~~		

	$\omega=0$	$0<\omega<2$	$\omega=2$	$2<\omega<\infty$	$\omega=\infty$
$I(G(j\omega))$	0	—	$-\frac{1}{8}$	—	0
$Re(G(j\omega))$	$\frac{1}{4}$	+	0	—	0

btw BC $s=Re^{j\theta}$

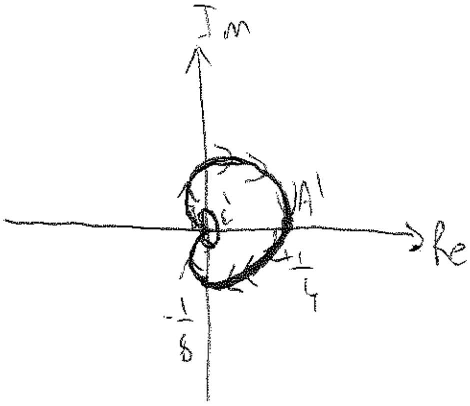
$$G(s)\Big|_{s=j\omega} = \frac{H(s)}{K}\Big|_{s=j\omega} = \frac{1}{(Re^{j\theta}+2)^2} \approx \frac{1}{R^2 e^{j2\theta}} = r e^{-2\theta}$$

$\theta=90 \rightarrow -2\theta=-180$

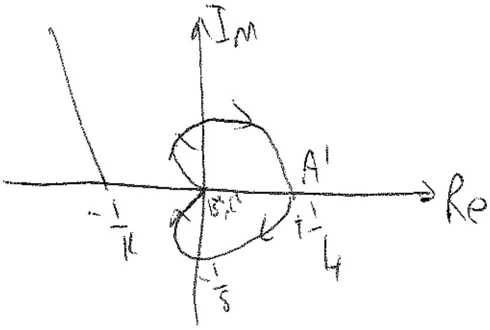
$\theta=45 \rightarrow -2\theta=-90$

$\theta=0 \rightarrow -2\theta=0$

Obtained contour



if $-\frac{1}{K} < 0$



number of encirclement of contour in clockwise direction

$N=0$

$N = Z - P$

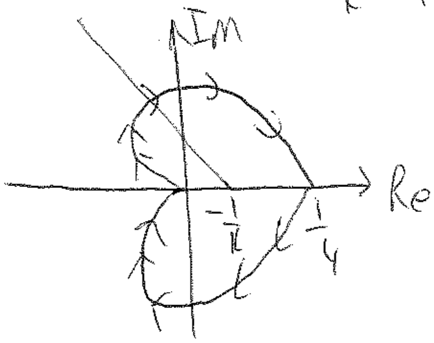
open loop poles in ORHP (open right-half plane)

$N=0=Z-P$

$P=0 \quad N=Z-P \quad 0=Z-0 \quad Z=0$ stable

$-\frac{1}{K}<0 \Rightarrow 0<\frac{1}{K} \quad K>0$

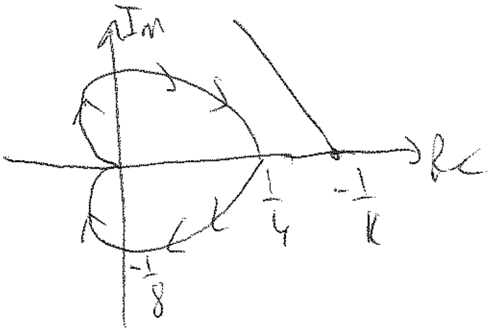
if $0<-\frac{1}{K}<\frac{1}{4}$



$-\frac{1}{K}<\frac{1}{4} \quad -\frac{1}{4}<\frac{1}{K} \quad -4>K$

$N=Z-P$

$1=Z-P^{\nearrow 0} \rightarrow Z=1$ (unstable) if $\frac{1}{4} < -\frac{1}{K}$



$\frac{1}{K} < -\frac{1}{4}$ $\quad K > -4$

$N = z - p$

$0 = z - 0$

$z = 0$ (stable)

Result

if ($K > 0$ or $K > -4$) meaning $K > -4$ the system is stable

if $K < -4$ the system is unstable

Q-2

$$G(s) = \frac{5}{s(s+2)}$$

$$G(j\omega) = \frac{5}{j\omega(j\omega+2)} \qquad |G(j\omega)| = \frac{5}{\omega\sqrt{\omega^2+4}} \qquad \angle G(j\omega) = -90 - \tan^{-1}\frac{\omega}{2}$$

$\omega_c \rightarrow$ gain crossover frequency $\quad |G(j\omega_c)| = 1 = \dfrac{5}{\omega_c\sqrt{\omega_c^2+4}}$

$$\omega_c\sqrt{\omega_c^2+4} = 5 \qquad \omega_c^2(\omega_c^2+4) = 25 \qquad \omega_c = a \qquad a(a+4) = 25$$

$$a^2 + 4a - 25 = 0 \qquad a = \frac{-4 \mp \sqrt{16-100}}{2} = \sqrt{29} - 2 \qquad a = \omega_c^2$$

$$\omega_c = \sqrt{\sqrt{29}-2}$$

$$\phi_m = \angle G(j\omega)\Big|_{\omega=\omega_c} + 180 = -90 - \tan^{-1}\frac{\sqrt{\sqrt{29}-2}}{2} + 180$$

$$\boxed{\phi_m = 90 - \tan^{-1}\frac{\sqrt{\sqrt{29}-2}}{2}}$$

Q-3

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \overset{A}{\begin{bmatrix}1&1\\1&0\end{bmatrix}}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u \qquad y=\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

(a) Find controllability matrix $Q_C = \begin{bmatrix}B & AB\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}$

$\text{rank}(Q_C) = 2$ (meaning $\det(Q_C) \neq 0$) system is completely controllable

(b) Find observability matrix $Q_O = \begin{bmatrix}C\\CA\end{bmatrix} = \begin{bmatrix}1&0\\1&1\end{bmatrix}$

$\text{rank}(Q_O) = 2$ (meaning $\det(Q_O) \neq 0$) system is completely observable

(c) Find $sI - A = \begin{bmatrix}s-1 & -1\\ -1 & s\end{bmatrix}$ $\quad \det(sI-A) = s(s-1) - (-1)(-1)$

$= s^2 - s - 1$

$$s^2 - s - 1 = 0 \Rightarrow s_{1,2} = \frac{+1 \mp \sqrt{1+4}}{2} = \frac{1 \mp \sqrt{5}}{2}$$

(d) Since the system is completely controllable by feedback we can relocate all the pole locations as we desire

using state feedback

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\left(\begin{bmatrix}k_1 & k_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + r\right)$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\left(\begin{bmatrix}1&1\\1&0\end{bmatrix}+\begin{bmatrix}0&0\\k_1&k_2\end{bmatrix}\right)}_{\tilde{A}}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix} r$$

$$\tilde{A} = \begin{bmatrix} 1 & 1 \\ 1+k_1 & k_2 \end{bmatrix}$$

$$sI - \tilde{A} = \begin{bmatrix} s-1 & -1 \\ -(1+k_1) & s-k_2 \end{bmatrix}$$

$$\det(sI - \tilde{A}) = (s-1)(s-k_2) - (-1)(-(1+k_1))$$
$$= s^2 + (-1-k_2)s + k_2 - 1 - k_1$$

$$s^2 + (-1-k_2)s + k_2 - 1 - k_1 = (s+3)^2 = s^2 + 6s + 9$$

$$-1-k_2 = 6 \qquad k_2 - 1 - k_1 = 9$$
$$k_2 = -7 \longrightarrow 7 - 1 - k_1 = 9 \qquad k_1 = -3$$

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \end{bmatrix}$$